

DISTINGUISHED REPRESENTATIONS OF $GL(n, \mathbb{C})$

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ABSTRACT. Let V be a $GL_n(\mathbb{R})$ -distinguished, irreducible, admissible representation of $GL_n(\mathbb{C})$. We prove that any continuous linear functional on V , which is invariant under the action of the real mirabolic subgroup, is automatically $GL_n(\mathbb{R})$ -invariant.

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1. INTRODUCTION

Let $G = GL(n, \mathbb{C})$, $H = GL(n, \mathbb{R})$ and $P \leq H$ be the standard mirabolic subgroup, i.e., the group consisting of matrices with last row equals to $(0, 0, \dots, 0, 1)$. We identify $H \backslash G$ with the space of matrices

$$X = \left\{ x \in G \mid x \cdot \bar{x} = I_n \right\},$$

via the isomorphism $Hg \mapsto \bar{g}^{-1} \cdot g$ (for the proof of the surjectivity of the map see Lemma 2.1). The group G acts on X by the twisted conjugation, $x\rho(g) := \bar{g}^{-1}xg$. For a topological vector space V , we denote by V^* the topological dual of V , i.e., the space of all continuous maps from V to \mathbb{C} . Let

$$\mathcal{D}(X) = C_c^\infty(X)^*$$

be the space of distributions on X . The space $\mathcal{D}(X)$ is a topological space with the standard topology ($T_n \rightarrow T$ if $T_n(f) \rightarrow T(f)$ for every $f \in C_c^\infty(X)$.)

In this paper we work with the category of the admissible smooth Fréchet representations of moderate growth, see [Wal, Section 11.5], see also [AGS, Section 2.1].

Let (π, V) be a representation of G . The representation (π, V) is called H -distinguished if there exists a non-zero continuous linear map $L : V \rightarrow \mathbb{C}$, such that

$$L(\pi(h)v) = L(v) \quad \forall v \in V, h \in H.$$

We denote the space of all such linear maps by $(V^*)^H$.

Let V, W be two representations in our category. Then we denote by $V \hat{\otimes} W$ the completed tensor product with the projective topology (this is the π -topology in [T, Definitions 43.2 and 43.5]).

We prove the following theorem

Theorem 1.1. *Let (π, V) be an irreducible, admissible H -distinguished representation of G . Then*

$$(V^*)^P = (V^*)^H.$$

The proof is based on section 3 of the article of Offen [O], see also [Ok]. It is splitted into two steps as follows.

Step 1: Reduction of the problem to a question on a single symmetric space X , defined above. More specifically, we prove that it is enough to show that $\mathcal{D}(X)^P = \mathcal{D}(X)^H$.

Step 2: We prove

Proposition 1.2. $\mathcal{D}(X)^P = \mathcal{D}(X)^H$.

Let $\xi \in \mathcal{D}(X)^P$. Note that the transpose acts on X . Indeed, let $x \in X$. Then $x\bar{x} = I_n$. Let us take the transpose and bar on both sides of the last equality. Then we obtain ${}^t(x)({}^t\bar{x}) = I_n$. Since the image of P under the map $g \mapsto {}^t\bar{g}^{-1}$ is tP and since ${}^t(x\rho(g)) = {}^t x\rho({}^t\bar{g}^{-1})$, we obtain that $\xi \in \mathcal{D}(X)^{{}^tP}$.

In order to prove Proposition 1.2 we use an idea that was suggested by Gourevitch (in personal communication). Actually, we prove a more general result, namely the following

Proposition 1.3. *Let $\xi \in \mathcal{D}(X)^{GL_{n-1}(\mathbb{R})}$. Then ${}^t\xi = \xi$, i.e., ξ is invariant under the transposition.*

Note that Proposition 1.3 implies Proposition 1.2. Indeed, let $\xi \in \mathcal{D}(X)^P$. Since $GL_{n-1}(\mathbb{R}) < P$ we have $\xi \in \mathcal{D}(X)^{GL_{n-1}(\mathbb{R})}$. Thus, by Proposition 1.3, the distribution ξ is invariant under the transposition, and hence $\xi \in \mathcal{D}(X)^{{}^tP}$. But H is generated by groups P and tP . Hence $\xi \in \mathcal{D}(X)^H$.

This result is important since it allows us to prove that the following two linear forms on the Whittaker model of an irreducible, generic, and unitarizable distinguished representation (π, V) defined by

$$W \mapsto \int_{U_n(\mathbb{R}) \backslash P_n(\mathbb{R})} W(p) dp,$$

and

$$W \mapsto \int_{U_n(\mathbb{R}) \backslash P_n(\mathbb{R})} W \left(\begin{pmatrix} 0 & 1 \\ I_{n-1} & 0 \end{pmatrix} p \right) dp$$

are $GL_n(\mathbb{R})$ -invariant. Here, $U_n(\mathbb{R})$ is the group of all upper triangular unipotent matrices of size $n \times n$.

The last result, combined with certain multiplicity one theorems, allows us to recover properties of epsilon factors of distinguished representations.

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2. NOTATION AND PRELIMINARIES

We begin with the following simple lemma, which states the surjectivity of the map $Hg \mapsto \bar{g}^{-1} \cdot g$.

Lemma 2.1. *Let $g \in G$ be a matrix such that $g \cdot \bar{g} = I_n$, where I_n is the identity matrix of size $n \times n$. Then there exists a matrix $h \in G$ such that $g = \bar{h}^{-1} \cdot h$.*

Proof. Let $\lambda \in \mathbb{C}$ be a scalar, define $h = \bar{\lambda}I + \lambda g$. One easily checks that

$$\bar{h} \cdot g = \lambda g + \bar{\lambda} \cdot I = h.$$

Hence, if h is invertible we get $g = \bar{h}^{-1} \cdot h$. Since λ is arbitrary, we can choose it in a way, so that $-\frac{\bar{\lambda}}{\lambda}$ is not an eigenvalue of g . This finishes the proof. \square

Consider an action of the 2-element group \mathbb{Z}_2 on H given by: $h \mapsto {}^t h^{-1}$. It induces the semidirect product

$$\tilde{H} := H \rtimes \mathbb{Z}_2.$$

Define a character χ on \tilde{H} by $\chi(g, n) := (-1)^n$.

We view the group $GL_{n-1}(\mathbb{R})$ as a subgroup of H - i.e. matrices of the form

$$\begin{pmatrix} A_{(n-1) \times (n-1)} & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly, define $\widetilde{GL_{n-1}(\mathbb{R})} := GL_{n-1}(\mathbb{R}) \rtimes \mathbb{Z}_2$. Clearly, this is a subgroup of \tilde{H} .

In the first stage of the proof we will use the following result ([AGS, Theorem A.0.3])

Theorem 2.2. (Frobenius descent) *Let G be a Lie group, assume G acts on two smooth manifolds X and Y , where the action on Y is transitive. Denote the actions by ρ_1 and ρ_2 respectively. Let $p : X \rightarrow Y$ be a smooth surjective map that commutes with the action of G , i.e., for all $g \in G$ the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\rho_1(g)} & X \\ \downarrow p & & \downarrow p \\ Y & \xrightarrow{\rho_2(g)} & Y \end{array}$$

For every $y \in Y$, let $G_y := \text{Stab}_G(y)$, the stabilizer of y in G . Then

$$\mathcal{D}(X)^G = \mathcal{D}(p^{-1}(y))^{G_y}.$$

Let (π, V) be a continuous representation of the Lie group G on a Fréchet space V . Define the space $V^{(1)}$ of differentiable vectors in V to be the set of all $v \in V$ such that the derivative

$$\frac{d}{dt} \pi(\exp(tx))v \Big|_{t=0}$$

exists for all $x \in \text{Lie}(G)$, where $\text{Lie}(G)$ is the Lie algebra of G . The resulting vector is denoted by $\pi(x)v$.

We now define inductively for $n \in \mathbb{N}$

$$V^{(n)} = \{v \in V^{(n-1)} \mid \pi(x)v \in V^{(n-1)} \text{ for all } x \in \text{Lie}(G)\}.$$

Define

$$V^\infty = \bigcap_{n=1}^{\infty} V^{(n)}.$$

Let $C_c^\infty(G)$ be the space of smooth compactly supported functions on G .

The following two theorems will be used in lemma 3.1.

Theorem 2.3. ([C], *Dixmier-Maliavin*) *Let G be a Lie group and (π, V) a continuous representation of G on a Fréchet space. Then every $v \in V^\infty$ may be represented as a finite linear combination*

$$v = \sum \pi(f_k)v_k,$$

where $f_k \in C_c^\infty(G)$ and $v_k \in V^\infty$, for all k .

Theorem 2.4. ([T, Thm 51.6,(c)]) *Let $K \subseteq \mathbb{R}^m$ and $L \subseteq \mathbb{R}^n$ be compact sets. Then*

$$C_c^\infty(K) \hat{\otimes} C_c^\infty(L) \cong C_c^\infty(K \times L).$$

Corollary 2.5. *Let $G = GL_n(\mathbb{C}) \subseteq \mathbb{R}^{2n^2}$. Then*

$$C_c^\infty(G \times G) \subseteq C_c^\infty(G) \hat{\otimes} C_c^\infty(G).$$

Proof. Let $f \in C_c^\infty(G \times G)$. Then $\text{supp}(f)$ is a compact set in $G \times G$, namely, there exist compact sets $K \subseteq \mathbb{R}^{2n^2}$ and $L \subseteq \mathbb{R}^{2n^2}$, such that $\text{supp}(f) \subseteq K \times L$. Hence,

$$f \in C_c^\infty(K \times L) \cong C_c^\infty(K) \hat{\otimes} C_c^\infty(L).$$

But $C_c^\infty(K) \hat{\otimes} C_c^\infty(L) \subseteq C_c^\infty(G) \hat{\otimes} C_c^\infty(G)$ and we obtain that $f \in C_c^\infty(G) \hat{\otimes} C_c^\infty(G)$. \square

Let M be a smooth manifold. Denote by $S(M)$ the Fréchet space of Schwartz functions on M . Denote by

$$S^*(M) := (S(M))^*$$

the space of Schwartz distributions on M .

In the second part of the proof Theorem 1.1, we use the following theorem of Aizenbud and Gourevitch, see further [AG, Propositions 3.1.3, 3.1.4 and Theorem A], see also [SZ].

Theorem 2.6. *Any $GL_n(\mathbb{R})$ -invariant distribution on $M_{n+1}(\mathbb{R})$ is invariant under the transposition.*

In fact, Theorem 2.6 implies that $\mathcal{D}(M_{n+1}(\mathbb{R}))^{\widetilde{GL_n(\mathbb{R})}, \chi} = 0$. Proposition 3.1.5 in [AG] together with Theorem 3.1.1 imply that

$$S^*(M_{n+1}(\mathbb{R}))^{\widetilde{GL_n(\mathbb{R})}, \chi} = 0.$$

It follows that a $GL_n(\mathbb{R})$ -invariant Schwartz distribution on $M_{n+1}(\mathbb{R})$ is also invariant under the transposition. Then the implication

$$(S^*(M_{n+1}(\mathbb{R}))^{\widetilde{GL_n(\mathbb{R})}, \chi} = 0) \Rightarrow (\mathcal{D}(M_{n+1}(\mathbb{R}))^{\widetilde{GL_n(\mathbb{R})}, \chi} = 0),$$

which follows from [AG2, Theorem 4.0.2], implies that a $GL_n(\mathbb{R})$ -invariant distribution on $M_{n+1}(\mathbb{R})$ is also invariant under the transposition.

3. PROPOSITION 1.2 IMPLIES THEOREM 1.1

Lemma 3.1. *Let (π, V) be an irreducible, admissible representation of G . Let $(\tilde{\pi}, \tilde{V})$ be the contragredient of (π, V) . Then there exists a morphism of $G \times G$ -modules with a dense image*

$$A_\pi : C_c^\infty(G) \rightarrow V \hat{\otimes} \tilde{V}.$$

Corollary 3.2. *The dual map*

$$A_\pi^* : V^* \hat{\otimes} \tilde{V}^* \rightarrow \mathcal{D}(G)$$

is an injective morphism of $G \times G$ -modules.

Proof. The proof follows [B, pp. 76-77]. We define the map A_π as a composition

$$A_\pi : C_c^\infty(G) \rightarrow \text{Hom}(V, V) \simeq V \hat{\otimes} V^*,$$

where the isomorphism $\text{Hom}(V, V) \simeq V \hat{\otimes} V^*$ follows from the standard theory of nuclear Fréchet spaces, see, for example, [AG, Lemma A.0.8].

The map $C_c^\infty(G) \rightarrow \text{Hom}(V, V)$ is given by

$$f \mapsto (v \mapsto \pi(f)v := \int_G f(g)(\pi(g)v)dg).$$

Since $f \in C_c^\infty(G)$, it follows that the image of A_π is contained in $(V \hat{\otimes} V^*)^\infty$. By the Dixmier-Malliavin theorem (Theorem 2.3), the space of smooth vectors in $V \hat{\otimes} V^*$ is a subspace of $V \hat{\otimes} \tilde{V}$. Indeed, let $v \in (V \hat{\otimes} V^*)^\infty$. Then by the Dixmier-Malliavin theorem,

$$v = \sum_{i=1}^k \pi(F_i)v_i,$$

where $v_i \in V \hat{\otimes} V^*$ and $F_i \in C_c^\infty(G \times G)$. Since $C_c^\infty(G \times G)(V \hat{\otimes} V^*) \subseteq V \hat{\otimes} \tilde{V}$, we have that $v \in V \hat{\otimes} \tilde{V}$.

To prove the inclusion $C_c^\infty(G \times G)(V \hat{\otimes} V^*) \subseteq V \hat{\otimes} \tilde{V}$ one can argue as follows. We know that for $f \in C_c^\infty(G)$, $v \in V$ and $v' \in V^*$ one has that $\pi(f)v \in V$ and $\pi(f)v' \in \tilde{V}$. By taking a tensor product of these maps we obtain a map

$$C_c^\infty(G) \hat{\otimes} V \hat{\otimes} C_c^\infty(G) \hat{\otimes} V^* \mapsto V \hat{\otimes} \tilde{V}.$$

We then have a map

$$(3.1) \quad (C_c^\infty(G) \hat{\otimes} C_c^\infty(G)) \hat{\otimes} (V \hat{\otimes} V^*) \mapsto V \hat{\otimes} \tilde{V}.$$

By Corollary 2.5, we have the following inclusion:

$$(3.2) \quad C_c^\infty(G \times G) \subseteq C_c^\infty(G) \hat{\otimes} C_c^\infty(G).$$

Combining 3.1 and 3.2, we obtain a non-zero morphism of $G \times G$ -modules

$$A_\pi : C_c^\infty(G) \rightarrow V \hat{\otimes} \tilde{V}.$$

Since $V \hat{\otimes} \tilde{V}$ is an irreducible $G \times G$ -module - see [AG, p.289] - we obtain that the map A_π has a dense image.

□

Theorem 3.3. *Suppose that*

$$\mathcal{D}(G)^{H \times P} = \mathcal{D}(G)^{H \times H}.$$

Then Theorem 1.1 holds.

Proof. The proof repeats almost verbatim [O, p. 177].

Let (π, V) be an irreducible, admissible H -distinguished representation. Then the contragredient representation $(\tilde{\pi}, \tilde{V})$ is also an irreducible, admissible H -distinguished representation. Indeed, by the theorem of Aizenbud, Gourevitch and Sayag, [AGS, Theorem 2.4.2], see also the theorem of Gelfand-Kazhdan in the p -adic case, [GK], the contragredient representation is H' -distinguished, where

$$H' = \{ {}^t h^{-1} | h \in H \}.$$

Since clearly $H' = H$, we get that the representation $(\tilde{\pi}, \tilde{V})$ is H -distinguished.

Take two non-zero linear forms $\lambda \in (\tilde{V}^*)^H$ and $\mu \in (V^*)^P$. Then, by Corollary 3.2,

$$0 \neq A_\pi^*(\mu \otimes \lambda) \in \mathcal{D}(G)^{P \times H}.$$

By the assumption, $A_\pi^*(\mu \otimes \lambda) \in \mathcal{D}(G)^{H \times H}$. Since A_π^* is an injective morphism of $G \times G$ -modules, it follows that $\mu \otimes \lambda \in (V^* \hat{\otimes} \tilde{V}^*)^{H \times H}$. Therefore $\mu \in (V^*)^H$. □

We see that in order to prove Theorem 1.1 it is enough to prove that

$$\mathcal{D}(G)^{H \times P} = \mathcal{D}(G)^{H \times H}.$$

We want to show that

$$(3.3) \quad \mathcal{D}(G)^{H \times P} \simeq \mathcal{D}(H \backslash G)^P \simeq \mathcal{D}(X)^P,$$

$$(3.4) \quad \mathcal{D}(G)^{H \times H} \simeq \mathcal{D}(H \backslash G)^H \simeq \mathcal{D}(X)^H.$$

These isomorphisms follow from the next lemma.

Lemma 3.4. *Let Q be a Lie group and let R be a closed Lie subgroup of Q . Then for any subgroup $Q' < Q$ we have*

$$(3.5) \quad \mathcal{D}(R \backslash Q)^{Q'} \simeq \mathcal{D}(Q)^{R \times Q'}.$$

Proof. We prove this by applying the Frobenius descent twice. Let

$$Y = R \backslash Q \times Q.$$

The group $Q \times Q'$ acts on Y by

$$\rho_1(q, q') \cdot (x, x') := (xq^{-1}, qx'q'^{-1}).$$

Take $Z = R \backslash Q$ with $Q \times Q'$ acting on Z by

$$\rho_2(q, q') \cdot x := xq^{-1}.$$

This action is transitive and we have a commutative diagram

$$(3.6) \quad \begin{array}{ccc} Y & \xrightarrow{\rho_1} & Y \\ \downarrow \phi & & \downarrow \phi \\ Z & \xrightarrow{\rho_2} & Z \end{array}$$

where $\phi(x, q') = x$ is the projection onto the first coordinate.
Let $x = Re_Q \in Z$ be the identity coset. The fiber above Re_Q is

$$\phi^{-1}(x) = \{x\} \times Q \simeq Q$$

and the stabilizer is $(Q \times Q')_x = R \times Q'$. We thus obtain

$$(3.7) \quad \mathcal{D}(Y)^{Q \times Q'} \simeq \mathcal{D}(Q)^{R \times Q'}.$$

The group $Q \times Q'$ acts on the group $W = Q$ by

$$\rho_3(q, q') \cdot x := qxq'^{-1}.$$

This action is transitive and we have a commutative diagram

$$(3.8) \quad \begin{array}{ccc} Y & \xrightarrow{\rho_1} & Y \\ \downarrow \psi & & \downarrow \psi \\ W & \xrightarrow{\rho_3} & W \end{array}$$

where $\psi(x, q') = q'$ is the projection onto the second coordinate.

The stabilizer of $x = e_Q \in W$ is $(Q \times Q')_x = \{(q, q') \in Q \times Q' \mid q = q'\} \simeq Q'$, and the fiber above x is

$$\psi^{-1}(x) = R \backslash Q \times x \simeq R \backslash Q.$$

We have that

$$(3.9) \quad \mathcal{D}(Y)^{Q \times Q'} \simeq \mathcal{D}(R \backslash Q)^{Q'}.$$

Combining (3.7) and (3.9), the result follows. \square

Note that H is indeed a closed subgroup of G . Thus, isomorphisms in (3.3) follow from Lemma 3.4 with $Q = G, R = H, Q' = P$ and isomorphisms (3.4) follow from this lemma with $Q = G, R = H, Q' = H$. To summarise, the main theorem follows by implying (3.3) and (3.4) to Proposition 1.2.

4. PROOF OF THE MAIN PROPOSITION 1.3

Before we begin the proof we shall introduce some notation.

Let

$$\mathfrak{X} = \left\{ x \in M_n(\mathbb{C}) : x + \bar{x} = 0 \right\} = i \cdot M_n(\mathbb{R}).$$

Thus, $x \mapsto ix$ is an isomorphism $M_n(\mathbb{R}) \simeq \mathfrak{X}$ of H -spaces, where H acts on both $M_n(\mathbb{R})$ and \mathfrak{X} by conjugation.

Let $\theta : \mathfrak{X} \rightarrow \mathbb{R}$,

$$\theta(x) = \det(I_n + x) \det(I_n - x) = \det(I_n + x) \overline{\det(I_n + x)}.$$

Clearly, θ is a continuous H -invariant map. The proof of Proposition 1.3 is given in two steps. We begin with the following linearization argument (the idea appeared in [AGS, p. 1511]).

Lemma 4.1. *Let*

$$\mathfrak{X}_0 = \theta^{-1}(\mathbb{R}^*) = \left\{ x \in \mathfrak{X} \mid \theta(x) \neq 0 \right\}.$$

Then $GL_{n-1}(\mathbb{R})$ acts on \mathfrak{X}_0 , and any $\xi \in \mathcal{D}(\mathfrak{X}_0)^{GL_{n-1}(\mathbb{R})}$ is invariant under the transposition.

Proof. The proof is an adaptation of Proposition 3.2.3 from [AGS] to our case. The idea is to take a distribution on \mathfrak{X}_0 and "extend" it to a distribution on \mathfrak{X} . By Theorem 2.6 we have that any $\xi \in \mathcal{D}(\mathfrak{X})^{GL_{n-1}(\mathbb{R})}$ is invariant under the transposition. Hence, we have $\mathcal{D}(\mathfrak{X})^{GL_{n-1}(\mathbb{R}), \chi} = 0$. We need to prove that $\mathcal{D}(\mathfrak{X}_0)^{GL_{n-1}(\mathbb{R}), \chi} = 0$.

Assume the contrary, and let $0 \neq \xi \in \mathcal{D}(\mathfrak{X}_0)^{GL_{n-1}(\mathbb{R}), \chi}$. Take $p \in \text{Supp}(\xi)$ and let $t = \theta(p)$. Note that $t \neq 0$. Let $f \in \mathcal{S}(\mathbb{R})$ be such that f vanishes in a neighborhood of zero and such that $f(t) \neq 0$. Consider $\xi' := (f \circ \theta) \cdot \xi$, i.e.,

$$\xi'(h) = \xi(h \cdot (f \circ \theta))$$

for all $h \in C_c^\infty(\mathfrak{X}_0)$. Then $\xi' \in \mathcal{D}(\mathfrak{X}_0)^{GL_{n-1}(\mathbb{R}), \chi}$. However, we can extend ξ' "by zero" to

$$\xi'' \in \mathcal{D}(\mathfrak{X})^{GL_{n-1}(\mathbb{R}), \chi} = 0.$$

Therefore $\xi' = 0$, and we obtained a contradiction! \square

Next, define a covering of X by open H -subspaces as follows. Note that if $\lambda \in \mathbb{C}^*$ is an eigenvalue of $x \in X$ then $\lambda\bar{\lambda} = 1$. For every such λ let

$$X_\lambda = \left\{ x \in X \mid \det(x - \lambda I_n) \neq 0 \right\}.$$

Note that $X - X_\lambda$ consists of all the matrices in X with λ as an eigenvalue. Since every matrix $x \in X$ has at most n different eigenvalues we obtain a finite covering of X by open H -subspaces

$$X = \bigcup_{i=1}^{n+1} X_{\lambda_i},$$

where $\lambda_1, \dots, \lambda_{n+1}$ are any $n+1$ distinct chosen numbers such that $\lambda_j \bar{\lambda}_j = 1$.

Next, we have an isomorphism of H -spaces \mathfrak{X}_0 and X_λ . The isomorphism is given by a Cayley transform

$$\xi_\lambda(x) = (x + \lambda I_n)(x - \lambda I_n)^{-1}.$$

The proof that this is indeed an isomorphism is given in [O], pp. 181-182. Our purpose is to prove that every distribution $\xi \in \mathcal{D}(X)$ which is $GL_{n-1}(\mathbb{R})$ -invariant is invariant under the transposition. This is equivalent to

$$(4.1) \quad \mathcal{D}(X)^{GL_{n-1}(\mathbb{R}), \chi} = 0.$$

The last equation now follows from the next simple lemma, which is a special case of [AG, Proposition 2.2.6].

Lemma 4.2. *Assume that a Lie group G acts on a smooth manifold X . Let*

$$X = \bigcup_{i=1}^k A_i$$

be a finite open covering, where A_i are G -invariant open subsets. Suppose $\mathcal{D}(A_i)^{G, \chi} = 0$ for all $i = 1, \dots, k$. Then

$$\mathcal{D}(X)^{G, \chi} = 0.$$

This completes the proof of Proposition 1.2 and therefore the proof of Theorem 1.1.

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